

ATNr: Allometric trophic models in R

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November 4, 2022

This document presents the mathematical definitions of the different models available in *ATNr*. *ATN* stands for Allometric Trophic Models. This name refers to the fact that these models describe the dynamic of populations that interact trophically with interaction strength determined by biological rates (like attack rate, handling time,...) derived from allometric relationships (i.e. body mass relationships). Since the seminal work from Yodzis and Innes in 1992 ([Yodzis and Innes, 1992]), different implementations were developed through time and the *ATNr* package propose three versions that are classically used in food web studies. Despite their differences, all *ATNr* models describe the biomass dynamic of species by estimating their growth rates at different point in time using a similar set of hypotheses:

- Growth rates of species are positively affected by what they consume (i.e. the models are based on energetic transfers between a resource and its consumer)
- Growth rates of species are negatively affected by consumers that feed upon them and by metabolic expenses.

Overall, the growth rate of non basal species over time can be formalised by a set of differential equations:

$$\frac{dB_i}{dt} = B_i \sum_j F_{ji}e_j - \sum_j B_j F_{ij} - X_i B_i, \quad (1)$$

where $\frac{dB_i}{dt}$ is the growth rate of the biomass of species i at a point in time t , B_i is the biomass of species i , F_{ij} is the per capita feeding rate of species j

on species i ($F_{ij} = 0$ if j does not feed on i), e_i is the assimilation efficiency of resource i when consumed and X_i is the per gram metabolic rate of species i . The growth rate of basal species is defined as:

$$\frac{dB_i}{dt} = r_i G_i B_i - \sum_j B_j F_{ij} - X_i B_i, \quad (2)$$

where r_i is the mass-specific growth rate of species i , G_i its net growth rate.

The three different versions of ATN models proposed in the package all derive from this set of equations, and only differ by how the feeding rate (i.e. species functional response) F_{ij} of non basal species and net growth rate of basal species G_i are calculated. For instance, a major difference in the calculation of G_i among models is whether or not the dynamic of the nutrient pool is considered.

Here we present the formulation of feeding rate and net growth rate used by the different models and depict how the different variables are usually defined by proposing a default parametrisation based on what is used in the literature. As a convention, for all parameters that depend on both resources and consumers, like F_{ij} , the first index refers to the resource and the second to the consumer. In other words, double subscripts read as “*from a resource to a consumer*”. This matches the data structure used in the package, where all matrices are defined such as rows represent resources and columns represent consumers.

1 Unscaled version

This version implements the model as in [Binzer et al., 2016]. It does not scale the time of the biological rate according to the growth rate of the smallest basal species, explaining its denomination (see section 2 for the scaled version). This model also does not include nutrient dynamics (see section 3).

1.1 functional response

In this version, the functional response F_{ij} describing the feeding rate of consumer j on resource i is written as:

$$F_{ij} = \frac{a_{ij} B_i^q}{1 + c_j B_j + \sum_k h_{kj} a_{kj} B_k^q}. \quad (3)$$

Here, B_i is the biomass of species i , a_{ij} is the attack rate of j on i , q is the hill exponent determining the shape of the functional response (for type II, $q = 1$; for type III, $q \in]1, 2]$), c sets the interference competition (the proportion of time that a consumer spends encountering con-specifics) and h_{ij} is the handling time of consumer j on resource i .

1.2 growth rate

The net growth rate G_i of basal species i is defined as:

$$G_i = 1 - \frac{s_i}{K_i}, \quad (4)$$

where K_i is the carrying capacity of species i and s_i depends on the ratio between plants inter- and intra-specific competition for resources α_{ij} :

$$s_i = \sum_j \alpha_{ij} B_j \quad (5)$$

The diagonal elements α_{ii} define intraspecific competition and the off-diagonal elements interspecific competition. The identity matrix ($\alpha_{ii} = 1, \alpha_{ij} = 0 \forall i \neq j$) corresponds to the model for which each basal species has its own resource (therefore, adding new basal species increases the total amount of resources available in the system). The matrix of ones ($\alpha_{ij} = 1 \forall i, j$) corresponds to a scenario where all basal species share the same resource pool.

Overall, this equation defines a saturating response: when the species biomass is low, net growth rate approaches 1. When the biomass is approaching the carrying capacity, the net growth rate tend to 0.

1.3 default parametrisation and units

A summary of the above parameters and their derivation can be found in Tables 1 and 2.

Variable	Mathematical expression	Name in the package	Definition
a_{ij}	$a_{ij} = a_0 m_i^{a_1} m_j^{a_2} e^{\frac{E_a}{kT} \frac{T_0 - T}{T_0}}$	\$a	Attack rate
h_{ij}	$h_{ij} = h_0 m_i^{h_1} m_j^{h_2} e^{\frac{E_h}{kT} \frac{T_0 - T}{T_0}}$	\$h	Handling time
c_i	Free parameter	\$c	Interference competition
K_i	$K_i = k_0 m_i^{k_1} e^{\frac{E_k}{kT} \frac{T_0 - T}{T_0}}$	\$K	Carrying capacity
r_i	$r_i = r_0 m_i^{r_1} e^{\frac{E_r}{kT} \frac{T_0 - T}{T_0}}$	\$r	Maximum growth rate
X_i	$X_i = x_0 m_i^{x_1} e^{\frac{E_x}{kT} \frac{T_0 - T}{T_0}}$	\$X	Metabolic rate
q_i	Free parameter	\$q	Hill exponent (functional response type)

Table 1: Parameter used for the unscaled model.

Variable (units)	parameter used	values
a_{ij} ($m^2 \cdot s^{-1}$)	a_0	$e^{-13.1}$
	a_1	0.25
	a_2	-0.8
	E_a	-0.38
h_{ij} (s)	h_0	$e^{9.66}$
	h_1	-0.45
	h_2	0.47
	E_h	0.26
c_i ($m^2 \cdot s^{-1}$)	c	0.8
K_i ($g \cdot m^{-2}$)	k_0	Free parameter
	k_1	0.28
	E_k	0.71
r_i ($g \cdot m^{-2}$)	r_0	$e^{-15.68}$
	r_1	-0.25
	E_r	-0.84
X_i ($J \cdot s^{-1}$)	x_0	$e^{-16.54}$
	x_1	-0.31
	E_x	-0.69
q_i	-	0.2

Table 2: Parameter units and default values used by the package *ATNr* for the model as in [Binzer et al., 2016].

2 Scaled version

This model, which corresponds to the one implemented in the julia package `BioEnergeticFoodWebs` ([Delmas et al., 2017]) refers to the *scaled* version of the ATN model. It means that the time unit of the different biological rates used (like metabolic rate, maximum feeding rate,...) are scaled to the growth rate of the smallest basal species (more details in 2.3). This redefinition of the variables implies a redefinition of the equations describing the dynamics of species, which introduce the scaled metabolic rate x_i and the maximum feeding rate of species relative to their metabolic rate y_i :

$$\frac{dB_i}{dt} = B_i r_i G_i - x_i B_i - \sum_j x_j y_j B_j F_{ij} \quad (6)$$

$$\frac{dB_i}{dt} = x_i y_i B_i \sum_j F_{ji} e_j - x_i B_i - \sum_j x_j y_j B_j F_{ij}, \quad (7)$$

2.1 functional response

In this model, the functional response can be expressed as:

$$F_{ij} = \frac{w_{ij} B_i^q}{B_0^q + c_j B_j + \sum_k w_{kj} B_k^q}. \quad (8)$$

where y_i is the maximum feeding rate of species i relative to its mass-specific metabolic rate x_i . w_{ij} is j 's relative consumption rate when consuming i , such as $\sum_i w_{ij} = 1$. c_j is the interference competition factor. q is the hill exponent determining the shape of the functional response (type II while $q = 1$, type III when $q > 1$ and $q \leq 2$)

2.2 growth rate

The net growth rate of basal species is here defined in the same way as for the unscaled version of the model:

$$G_i = 1 - \frac{s_i}{K_i}, \quad (9)$$

where K_i is the carrying capacity of species i . s_i depends on the ratio between plants inter- and intra-specific competition for resources α_{ij} :

$$s_i = \sum_j \alpha_{i,j} * B_j \quad (10)$$

The diagonal elements α_{ii} define intraspecific competition and the off-diagonal elements interspecific competition. The identity matrix ($\alpha_{ii} = 1, \alpha_{ij} = 0$ for all (i, j)) correspond to the model for which each basal species has its own resource (therefore, adding new basal species increases the total amount of resources available in the system). The matrix of ones ($\alpha_{ij} = 1$ for all (i, j)) correspond to a scenario where all basal species shared the same resource pool.

Overall, this equation defines a saturating response: when the species biomass is low, net growth rate approaches 1. When the biomass is approaching the carrying capacity, the net growth rate tend to 0.

2.3 parametrisation and units

As in the previous models, the biological rates are based on allometric relationships (but here, with an exponent of -0.25). However, the rates used in the model correspond to scaled version of the natural biological rates. The scaling is done to express biological rates relative to the growth rate of the smallest basal species. Thereafter, and consistently with the notation used in the functional response, we use capital letters for the natural rates and small letters for their scaled versions. As before, the natural biological rates are defined as:

$$R_i = r_0 M_i^{-0.25} \quad (11)$$

and the same definitions hold for mass-specific X_i metabolic rate and maximum feeding rate Y_i :

$$X_i = x_0 M_i^{-0.25}, \quad (12)$$

$$Y_i = y_0 M_i^{-0.25}, \quad (13)$$

Then the scaling is done using the following transformations, assuming that the smallest basal species is species 1:

$$r_i = \frac{r_0 M_i^{-0.25}}{r_0 M_1^{-0.25}} = \frac{M_i^{-0.25}}{M_1^{-0.25}}, \quad (14)$$

$$x_i = \frac{x_0 M_i^{-0.25}}{r_0 M_1^{-0.25}} = \frac{x_0}{r_0} \left(\frac{M_i}{M_1} \right)^{-0.25}. \quad (15)$$

y_i is the maximum consumption rate of population i relative to its metabolic rate:

$$y_i = \frac{y_i}{x_i} = \frac{\frac{y_0 M_i^{-0.25}}{r_0 M_1^{-0.25}}}{\frac{x_0 M_i^{-0.25}}{r_0 M_1^{-0.25}}} = \frac{y_0}{x_0}. \quad (16)$$

The values associated to these parameters are presented in table 3.

Variable	parameter used	values	Variable name in the package
x_i	$\frac{x_0}{r_0}$	0.314	<code>\$X</code>
y_i	$\frac{a_y}{a_x}$	8	<code>\$max_feed</code>
K_p	K	10	<code>\$K</code>
q_i	-	1.2	<code>\$q</code>

Table 3: values and units of variables as set by the package default parametrization for the scaled version

3 Unscaled version with nutrient dynamic

For this model, described in [Schneider et al., 2016], the definition of the feeding rate is based on the unscaled version (i.e. like in [Binzer et al., 2016], described in section 1), with a slight difference in the way the attack rate is defined. It strongly departs from the models presented before in the way the net growth rate of plants is calculated: instead of considering a maximum carrying capacity for basal species, the growth rate of plants is determined by their interactions with a nutrient pool for which the dynamics of nutrient concentrations are explicitly modeled (using differential equations, such as for species biomass dynamics).

3.1 functional response

The functional response F_{ij} describing the feeding rate of consumer j on resource i is written as:

$$F_{ij} = \frac{w_{ij} b_{ij} B_i^q}{1 + c_j B_j + \sum_k w_{kj} h_{kj} a_{kj} B_k^q} \frac{1}{m_i}. \quad (17)$$

As before, q is the hill exponent determining the shape of the functional response (type II while $q = 1$, type III when $q > 1$ and $q \leq 2$), c sets the interference competition and h_{ij} is the handling time of consumer j on resource i . Here, the attack rate a_{ij} has been substituted by a resource specific capture coefficient b_{ij} (see section 3.3 on parameters for more details). w_{ij} is j 's relative consumption rate when consuming i , such as $\sum_i w_{ij} = 1$. m_i corresponds to the body mass of species i .

3.2 growth rate

The net growth rate of plant species G_i is not defined from a parameter corresponding to a carrying capacity but from concentrations of a set of nutrients for which the dynamic over time is explicitly modeled. when plants acquire nutrients from n different nutrient pools, G_i is defined as:

$$G_i = \min \left(\frac{N_1}{K_{1i} + N_1}, \dots, \frac{N_n}{K_{ni} + N_n} \right) \quad (18)$$

where k_{ni} determines the nutrient uptake efficiency of plant i on nutrient n . The smaller k_{ni} is, the more efficient plant i is to uptake nutrient n . N_n is the concentration of nutrient n , which dynamically changes over time and is described by another set of differential equations:

$$\frac{dN_n}{dt} = D(S_n - N_n) - v_{ni} \sum_i r_i G_i B_i. \quad (19)$$

Here, D is the global turnover rate that determines the rate by which the nutrients are refreshed. S_n is the maximal concentration of nutrient n . v_{ni} sets the relative content of nutrient n in plant i .

3.3 parametrisation and units

Variable	Mathematical expression	Name in the package	Definitions
L_{ij}	$L_{ij} = \left(\frac{m_j}{m_i R_{opt}} e^{1 - \frac{m_j}{m_i R_{opt}}} \right)$	\$L	Attack rate
b_{ij}	$b_{ij} = b_0 m_i^{b_1} m_j^{b_2} e^{E_b \frac{T_0 - T}{k T T_0}} L_{ij}$	\$b	Attack rate
h_{ij}	$h_{ij} = h_0 m_i^{h_1} m_j^{h_2} e^{E_h \frac{T_0 - T}{k T T_0}}$	\$h	Handling time
r_i	$r_i = m_i^{r_1} e^{E_r \frac{T_0 - T}{k T T_0}}$	\$r	Maximum growth rate
X_i	$X_i = x_0 m_i^{x_1} e^{E_x \frac{T_0 - T}{k T T_0}}$	\$X	Metabolic rate
K_{np}	Free parameter	\$K	Uptake efficiency
v_{np}	Free parameter	\$V	Plants relative nutrient content
S_n	Free parameter	\$S	Maximal level for nutrients
c	Free parameter	\$c	Interference competition
q_i	Free parameter	\$q	Hill exponent (functional response type)
D	Free parameters	\$D	Turnover rate of the nutrients

Table 4: variables used in the unscaled with nutrient version

Variable (units)	parameter used	values
b_{ij} ($m^2 \cdot s^{-1}$)	b_0	50
	b_1	$\mathcal{N}(0.15, 0.03)$
	b_2	$\mathcal{N}(0.47, 0.04)$
	E_a	-0.38
h_{ij} (s)	h_0	0.4
	h_1	$\mathcal{N}(-0.66, 0.02)$
	h_2	$\mathcal{N}(-0.48, 0.03)$
	E_h	0.26
c_i (s)	c	$\mathcal{N}(0.8, 0.2)$
K_{np} ($g \cdot m^{-2}$)	k	$\mathcal{U}(0.1, 0.2)$
v_{np} (unitless)	v	$\mathcal{U}(1, 2), \sum_n (v_{np} = 1)$
S_n ($g \cdot m^{-2}$)	s	$\mathcal{N}(10, 2)$
r_p (s^{-1})	r_1	-0.25
	E_r	-0.84
X_i ($J \cdot s^{-1}$)	x_0	0.138 (i is a plant)
	x_0	0.314 (i is an animal)
	x_1	-0.25
	E_x	-0.69
q_i	-	0.2

Table 5: values and units of variables as set by the package default parametrisation for the unscaled with nutrient version

References

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